Gravitational waves from inflation on the brane

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We discuss the evolution of gravitational waves in a brane-world cosmology embedded in five-dimensional anti-de Sitter spacetime. We show that during slow-roll inflation, modelled as a period of quasi-de Sitter expansion on the brane, there is a discrete normalizable massless graviton mode. There is a mass gap due to the expansion, above which there is a continuum of massive modes. Only the massless mode is 'light' compared with the Hubble scale during inflation, leading to the production of classical perturbations on large scales from vacuum fluctuations on small scales. We calculate the amplitude of these fluctuations at horizon-crossing and show that the standard four-dimensional result is recovered at low energies, but the amplitude of the perturbations is enhanced at high energies.

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I. INTRODUCTION

The idea that there may exist large extra dimensions, inspired by recent developments in M/string-theory, has recently attracted much attention. This possibility arises when one assumes that ordinary matter is confined to a brane, our 'Universe', within a higher-dimensional spacetime. In this spirit, Randall and Sundrum [1] proposed a five-dimensional model with a curved metric, for which they showed that the linearized gravitational interaction behaves effectively as standard four-dimensional gravity (plus correction terms), even though the fifth dimension may be large, possibly infinite.

Although appealing, these models have still to be confronted with observations. One line of investigation is to search for exotic effects in accelerators. Another, to which the present work belongs, is to explore the consequences of these models in cosmology. The first step, undertaken during the last year, has been to study the homogeneous and isotropic cosmologies of brane models, and to see whether they are compatible with cosmological evolution. The next step is to study the cosmological perturbations and confront them with the increasingly detailed observations of large scale structure and cosmic microwave background (CMB) anisotropies.

Cosmological perturbations in brane models are however more complicated to deal with than in the standard theory of cosmological perturbations. The general formalism has only very recently been investigated [2–6]. In the present work, our goal is to study the evolution of gravitational waves, which are not related directly to the matter perturbations in the brane, and are thus in principle easier to study. However, the equation of motion for gravitational waves is complicated because in general one must solve a two-variable partial differential equation, even after decomposition into Fourier modes. In the Randall-Sundrum (RS) scenario of a Minkowski background brane, the equation separates and may be solved exactly [1]. For an expanding background brane, this is in general no longer possible in coordinates in which the brane is fixed in the bulk. It turns out, however, that there is one case where it is possible: de Sitter evolution on the brane. Thus we are able to make quantitative predictions about the production and evolution of gravitational waves during slow-roll inflation on the brane. Scalar perturbations in this scenario have been previously investigated in [7]. As in the non-expanding solutions, the massless mode is decoupled from the massive modes and it is possible to avoid contamination of the massless graviton modes by massive modes on the brane during slow-roll inflation. If this were not the case, there would be potentially problematic production of massive graviton modes during inflation, as pointed out in [2].

II. COSMOLOGICAL TENSOR PERTURBATIONS

A. Background metric

Before addressing the cosmological perturbations, let us specify the background five-dimensional spacetime we are going to use. The metric is of the form

$$g_{AB}dx^A dx^B = -n^2(t,y)dt^2 + a^2(t,y)d\vec{x}^{\,2} + dy^2\,, \quad (1)$$

where $x^A = (x^\mu, x^5) = (t, x^i, y)$. In the following, a dot will denote $\partial/\partial t$ and a prime $\partial/\partial y$. Note that we have assumed, for simplicity, that the spatial three-surfaces are flat. Cosmological solutions can then be obtained [8] by solving the five-dimensional Einstein equations,

$$G_{AB} \equiv R_{AB} - \tfrac{1}{2} R g_{AB} = \kappa_5^2 T_{AB}\,, \tag{2}$$

where R_{AB} is the five-dimensional Ricci tensor and $R = g^{AB} R_{AB}$ its trace. The gravitational constant is related to the five-dimensional Planck scale, $\kappa_5^2 \equiv 8\pi/M_5^3$. The energy-momentum tensor on the right hand side can be decomposed into

$$T^A{}_B = -\kappa_5^{-2} \Lambda_5 \delta^A{}_B + \delta(y) S^A{}_B\,, \tag{3}$$

where we assume there is only a cosmological constant in the bulk, thus generalizing the RS scenario to cosmology. The term $S^A{}_B$ is the energy-momentum tensor of the matter localized on the brane ($y = 0$). Because of the symmetries of the metric in Eq. (1), this is necessarily of perfect-fluid form:

$$S^A{}_B = \mathrm{diag}\left(-\rho_\mathrm{b}, p_\mathrm{b}, p_\mathrm{b}, p_\mathrm{b}, 0\right), \tag{4}$$

where the energy density ρ_b and pressure p_b are functions only of time t. Also, because $T^0_5 = 0$, it follows from Einstein's equations that

$$\left(\frac{\dot a}{n}\right)' = 0, \tag{5}$$

which implies that, if one chooses t to coincide with the proper time on the brane, one gets

$$n = \frac{\dot a}{\dot a_o}\,. \tag{6}$$

The influence of the brane matter on the five-dimensional metric enters via the junction conditions,

$$[K_{\mu\nu}] = -\kappa_5^2 \left(S_{\mu\nu} - \tfrac{1}{3} S g_{\mu\nu}\right)\,, \tag{7}$$

where the square brackets denote the 'jump' across the brane ($[f] \equiv f(0^+) - f(0^-)$), and $K_{\mu\nu}$ is the extrinsic curvature tensor of the brane. This implies for the metric components [8]

$$\frac{[a']}{a_o} = -\frac{\kappa_5^2}{3}\rho_\mathrm{b}\,, \qquad \frac{[n']}{n_o} = \frac{\kappa_5^2}{3}\left(3p_\mathrm{b} + 2\rho_\mathrm{b}\right)\,, \tag{8}$$

where the subscript 'o' denotes the value taken on $y = 0$. Assuming in addition Z_2 symmetry, the jump is then simply twice the value on one side.

It is then possible to find, using Einstein's equations (2), a first integral that is analogous to the Friedmann equation, but which reads [9] (see also [10,11])

$$H_o^2 \equiv \frac{\dot a_o^2}{a_o^2} = \frac{\Lambda_5}{6} + \frac{\kappa_5^4}{36}\rho_\mathrm{b}^2 + \frac{\mathcal{C}}{a_o^4}\,, \tag{9}$$

where $\mathcal{C}$ is an integration constant which can be interpreted as arising from a five-dimensional tidal effect (see [11,3]). Combining Eq. (9) with the conservation equation

$$\dot\rho_\mathrm{b} + 3\frac{\dot a_o}{a_o}(\rho_\mathrm{b} + p_\mathrm{b}) = 0 \tag{10}$$

implied by Einstein's equations (2), which is the same as in standard cosmology, one can obtain the cosmological evolution in the brane.

In general, the scale factor $a_o(t)$, does not follow the conventional cosmological evolution [8]. However, standard cosmology can be recovered at low energies [12,9] in the particular case where the brane is endowed with a constant tension λ, so that

$$\rho_\mathrm{b} = \rho + \lambda\,, \quad p_\mathrm{b} = p - \lambda\,.$$

This allows one (when $\mathcal{C} = 0$) to regain the general relativistic Friedmann equation from Eq. (9) in the low-energy limit $\rho \ll \lambda$, by identifying

$$\kappa_4^2 \equiv \tfrac{1}{6}\kappa_5^4 \lambda\,, \tag{11}$$

where the effective four-dimensional Planck mass is given by $M_4^2 = 8\pi/\kappa_4^2$. Then Eq. (9) becomes

$$H_o^2 = \frac{\Lambda_4}{3} + \frac{\kappa_4^2}{3}\rho\left(1 + \frac{\rho}{2\lambda}\right) + \frac{\mathcal{C}}{a_o^4}\,, \tag{12}$$

where $\Lambda_4 = (\Lambda_5 + \frac{1}{6}\kappa_5^4\lambda^2)/2$. If the brane tension λ is adjusted so as to compensate for the (negative) bulk cosmological constant Λ_5 (so that $\Lambda_4 = 0$), we require

$$\kappa_5^2 \lambda = \sqrt{-6\Lambda_5}\,.$$

Finally, one can solve for the metric in the bulk, to get the explicit dependence of the metric on the coordinates y and t [9]. The bulk metric can be shown to be a Schwarzschild-anti de Sitter solution, where the Schwarzschild mass parameter is proportional to $\mathcal{C}$ [13]. For solutions with $\mathcal{C} = 0$, it follows that the bulk is anti-de Sitter, and thus conformally flat, while $\mathcal{C} \neq 0$ signals a non-conformally flat bulk.

B. Tensor perturbations

Linear perturbations in brane cosmology have been studied very recently [2–6]. In the present work, we will focus only on gravitational waves (or tensor modes in the usual terminology in cosmology [14]), and leave aside the scalar and vector modes. Gravitational waves can propagate independently of the matter content in the universe, and they are therefore simpler to study. Moreover, these modes were the first to be investigated in the non-cosmological (static) case [1].

Following [4], we will thus consider a perturbed metric of the form

$$ds^2 = -n^2 dt^2 + a^2\left(\delta_{ij} + E_{ij}\right) dx^i dx^j + dy^2\,, \tag{13}$$

where E_{ij} is transverse traceless, i.e., $\partial_i E^{ij} = 0$ and $E^i{}_i = 0$ (using the metric δ_{ij} to raise or lower the spatial indices). Note that these tensor perturbations are automatically gauge-invariant, i.e. are not affected by a coordinate change to first order.

The equation of motion for the tensor modes is then obtained from the linear perturbations of Einstein's equations. In the bulk, since there is only a cosmological constant, these reduce simply to

$$\delta R_{ij} = \tfrac{2}{3}\Lambda_5 E_{ij}\,, \tag{14}$$

which yields the following equation of motion [4]:

$$\begin{aligned}&\frac{1}{n^2}\ddot{E} + \frac{1}{n^2}\left(\frac{3\dot{a}}{a} - \frac{\dot{n}}{n}\right)\dot{E} + \frac{k^2}{a^2}E \\ &-E'' - \left(3\frac{a'}{a} + \frac{n'}{n}\right)E' = 0\,,\end{aligned} \tag{15}$$

where we have decomposed the general metric perturbation into Fourier modes:

$$E^i{}_j(t,\vec{x},y) = E(t,y;\vec{k})\,\exp(\mathrm{i}\vec{k}\cdot\vec{x})\,\hat{e}^i{}_j\,,$$

with $\hat{e}_{ij}$ a transverse traceless polarization tensor. We can rewrite Eq. (15) as

$$\left(\frac{a^3}{n}\dot{E}\right)^{\cdot} + k^2 anE - \left(a^3 nE'\right)' = 0\,.$$

These equations simplify dramatically if $n = a$ and $\dot{n} = 0 = \dot{a}$, which is the RS case, for which the equation separates and may be solved exactly [1]. For an expanding background brane, separation is in general no longer possible in coordinates in which the brane remains fixed in the bulk. It is always possible (for $\mathcal{C} = 0$) to write the general solution for linear perturbations about five-dimensional anti-de Sitter spacetime in a manifestly static coordinate system [6], but it is non-trivial to impose the boundary conditions at the brane, which is then moving.

The wave equation (15) is to be solved using the background form of $a(t,y)$, as given in Ref. [9]. The boundary condition at the brane follows from the linearly perturbed junction conditions given in Eq. (7):

$$E'_{ij}\Big|_{y=0^+} = \bar{\pi}_{ij}\,,$$

where $\bar{\pi}_{ij}$ is the tensor contribution of the anisotropic stress exerted by matter on the brane, which we will henceforth assume to be zero. Thus we set

$$E'|_o = 0\,. \tag{16}$$

However, we note that E'' in Eq. (15) remains nonzero on the brane, and we need to solve for the y-dependence of E.

In order to gain further insight, we can expand in the off-brane direction. Using the boundary conditions in Eqs. (8) and (16) and taking t as the brane proper time, we find that

$$\begin{aligned}a(t,y) &= a_o(t)\left[1 - \tfrac{1}{6}\kappa_5^2\{\rho(t)+\lambda\}|y| + O(y^2)\right]\,,\\ n(t,y) &= 1 + \tfrac{1}{6}\kappa_5^2\{3p(t)+2\rho(t)-\lambda\}|y| + O(y^2)\,,\\ E(t,y) &= E_o(t) + \tfrac{1}{2}E_2(t)y^2 + \tfrac{1}{6}E_3(t)|y|^3 + O(y^4)\,,\end{aligned}$$

where $E_n(t) = \partial^n E(t,0^+)/\partial y^n$. Then expanding Eq. (15) order by order yields

$$E_2 = \ddot{E}_o + 3H_o\dot{E}_o + \frac{k^2}{a_o^2}E_o\,, \tag{17}$$

$$\begin{aligned}E_3 = &-\tfrac{1}{2}\kappa_5^2(\rho+3p-2\lambda)E_2 \\ &+ \tfrac{1}{2}\kappa_5^2(\rho+p)\left[(5+3c_{\rm s}^2)H_o\dot{E}_o + 2\frac{k^2}{a_o^2}E_o\right]\,,\end{aligned} \tag{18}$$

$(\cdots)\quad ,$

where $c_{\rm s}^2 = \dot{p}/\dot{\rho}$. These equations determine the y-derivatives on the brane of the tensor perturbations, in terms of E_o. The 5-dimensional bulk equation (15) is replaced by the infinite sequence of equations (17), (18), $(\ldots)$, on the brane.

If $E_2 = 0$, i.e., if $E'' = 0$ on the brane, then Eq. (17) reduces to the standard general relativity equation for tensor perturbations, generalizing the RS zero-mode [1] to a cosmological context. Its solution $E_o(t)$ then fixes E_n, for $n \geq 3$, via the subsequent equations. This shows that $E = E_o(t)$, where $E_n = 0$ for all $n \geq 1$, is inconsistent in general, since E_o is subject to an infinite chain of constraints. In general the "zero-mode" is coupled to the higher-order terms.

However there is one important limit in which the "zero-mode" remains decoupled. On large scales, where we can neglect the k^2 term, the gravitational wave equation (15) always admits the homogeneous solution $E(t,y;k\to 0) = E_o = $ constant, which, of course, satisfies the boundary condition in Eq. (16). Thus the standard result from general relativity in four dimensions that the amplitude of tensor perturbations stays constant on super-horizon scales remains true in a brane-world cosmology*.

The other case in which terms involving E_0 drop out of Eq. (18) for E_3 and all higher E_n is when $\rho + p = 0$, corresponding to de Sitter expansion in the brane, and we will now consider this case in greater detail.

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*A similar result can be obtained for scalar metric perturbations corresponding to adiabatic density perturbations on large scales [7,15].

III. BULK GRAVITONS WITH A DE SITTER BRANE

A. Separable background

In order to solve for the y-dependence of the bulk gravitons and to study the time-dependence of the perturbations on the brane, we will consider separable solutions for the bulk metric, i.e., where the scale factor is of the form

$$a(t,y) = a_o(t)\mathcal{A}(y)\,, \quad \mathcal{A}(0) = 1\,. \tag{19}$$

It follows from Eq. (6) that

$$n = \mathcal{A}(y)\,, \tag{20}$$

and then the junction conditions in Eq. (8) immediately require that the stresses on the brane obey the equation of state $p_{\rm b} = -\rho_{\rm b}$ =constant, so that $p = -\rho$ = constant. This is the case for a constant scalar field on the brane, and it will be a good approximation to a potential-dominated scalar field rolling slowly down a sufficiently flat potential [7].

Substituting (19-20) into Einstein's equations shows that $\mathcal{C} = 0$, so that the bulk is conformally flat anti de Sitter spacetime. The generalized Friedmann equation (12) then shows that H_o =constant, so that $a_o(t) = \exp(H_o t)$. Thus a separable scale factor in the coordinates of Eq. (1) arises if and only if the induced metric on the brane is de Sitter (including the RS static case). Solving for the metric in the bulk then yields [16]

$$\mathcal{A}(y) = \cosh(\mu y) - \left[1 + \frac{\rho}{\lambda}\right]\sinh(\mu|y|)\,, \tag{21}$$

where

$$\mu = \frac{\kappa_4^2}{\kappa_5^2} = \frac{M_5^3}{M_4^2}\,. \tag{22}$$

Constant-y hypersurfaces correspond to exponentially expanding de Sitter slices for $\rho > 0$, giving a dS_4 slicing of AdS_5. The original RS solution with Minkowski spacetime on the brane (M_4 slicing of AdS_5) is recovered in the limit $\rho/\lambda \to 0$, when $\mathcal{A} \to \exp(-\mu|y|)$. At very high energies, $\rho \gg \lambda$, deviations from the RS solution will be significant.

The y-dependence of the scale factor can be conveniently rewritten in the form

$$\mathcal{A}(y) = \frac{H_o}{\mu}\sinh\mu(y_{\rm h} - |y|)\,, \tag{23}$$

where $y = \pm y_{\rm h}$ are Cauchy horizons ($g_{00}(\pm y_{\rm h}) = 0$) [16], with

$$y_{\rm h} = \frac{1}{\mu}\coth^{-1}\left(1 + \frac{\rho}{\lambda}\right)\,. \tag{24}$$

In the RS model, $y_{\rm h} \to \infty$. It is useful to introduce the conformal bulk-coordinate $z = \int dy/\mathcal{A}(y)$:

$$z = \mathrm{sgn}(y) H_o^{-1}\ln\left[\coth\tfrac{1}{2}\mu(y_{\rm h} - |y|)\right]\,. \tag{25}$$

The Cauchy horizon is now at $|z| = \infty$, and the brane is located at $z = \pm z_{\rm b}$, with

$$z_{\rm b} = \frac{1}{H_o}\sinh^{-1}\frac{H_o}{\mu}\,. \tag{26}$$

The line element, Eq. (1), becomes

$$ds^2 = \mathcal{A}^2(z)\left[-dt^2 + dz^2 + \mathrm{e}^{2H_o t} d\vec{x}^{\,2}\right]\,, \tag{27}$$

where

$$\mathcal{A}(z) = \frac{H_o}{\mu\sinh(H_o|z|)}\,. \tag{28}$$

The (t,z) coordinates are well-suited to describing the causal structure of the bulk inside the Cauchy horizon, which is conformal to Minkowski spacetime with the region $|z| < z_{\rm b}$ cut out and the $z = \pm z_{\rm b}$ hypersurfaces identified. (The brane tension generates the discontinuity in the extrinsic curvature either side of $z = \pm z_{\rm b}$.) In particular this will enable us to treat the bulk gravitons as perturbations evolving in a 2-D Minkowski spacetime.

B. Separation of perturbation variables

In the dS_4 slicing of AdS_5, the gravitational wave equation (15) reduces to

$$\ddot{E} + 3H_o\dot{E} + \frac{k^2}{a_o^2}E = \mathcal{A}^2 E'' + 4\mathcal{A}\mathcal{A}'E'\,, \tag{29}$$

which can then be separated into eigenmodes of the time-dependent equation on the brane and the off-brane equation:

$$E(t,y;\vec{k}) = \int dm\,\varphi_m(t;\vec{k})\mathcal{E}_m(y)\,,$$

where

$$\ddot{\varphi}_m + 3H_o\dot{\varphi}_m + \left[m^2 + \frac{k^2}{a_o^2}\right]\varphi_m = 0\,, \tag{30}$$

$$\mathcal{E}_m'' + 4\frac{\mathcal{A}'}{\mathcal{A}}\mathcal{E}_m' + \frac{m^2}{\mathcal{A}^2}\mathcal{E}_m = 0\,. \tag{31}$$

We recover the RS solutions in the limit $H_o \to 0$, $\rho \to 0$, in which case $\varphi_m = \exp(\pm \mathrm{i}\omega t)$, with $\omega^2 = k^2 + m^2$, and $\mathcal{E}_m$ can be given in terms of Bessel functions of order 2 [1].

For $H_o > 0$, the time-dependent function $\varphi_m(t;\vec{k})$ obeys the wave equation (30) for a massive scalar field in de Sitter 4-spacetime. If we write $u_m = a_o\varphi_m$ and work

in terms of the conformal time $\eta = -1/(a_o H_o)$, the wave equation can be rewritten as

$$\frac{d^2 u_m}{d\eta^2} + \left[k^2 - \frac{2-(m^2/H_o^2)}{\eta^2}\right] u_m = 0\,,$$

whose general solution is given by

$$u_m(\eta;\vec{k}) = \sqrt{-k\eta}\, B_\nu(-k\eta)\,, \quad \nu^2 = \frac{9}{4} - \frac{m^2}{H_o^2}\,,$$

where B_ν is a linear combination of Bessel functions of order ν. The solutions oscillate at early-times/small-scales for all m, with an approximately constant amplitude while they remain within the de Sitter event horizon ($k \gg a_o H_o$). 'Heavy modes', with $m > \frac{3}{2}H_o$, continue to oscillate as they are stretched to super-horizon scales, but their amplitude rapidly decays away, $|u_m^2| \propto a_o^{-3}$. But for 'light modes' with $m < \frac{3}{2}H_o$, the perturbations become over-damped at late-times/large-scales ($k \ll a_o H_o$), and decay more slowly: $|u_m^2| \propto a_o^{2\nu-3}$. In particular the zero-mode, φ_o, approaches a constant value at late times.

The general solution for the y-dependence of the zero-mode ($m = 0$) in a de Sitter cosmology is

$$\mathcal{E}_o = C_1 + C_2 \mathrm{sgn}(y) \coth\mu(y_\mathrm{h}-|y|)\left[3-\coth^2\mu(y_\mathrm{h}-|y|)\right]\,,$$

where C_a are constants to be determined by the boundary conditions. The junction condition in Eq. (16) requires $C_2 = 0$, and thus $\mathcal{E}_o =$ constant. The metric perturbation in Eq. (13) for the zero-mode is then given by

$$\begin{aligned} h_{ij}(t,\vec{x},y) &= a_o^2(t)\mathcal{A}^2(y)E_{ij}(t,\vec{x},y)\,,\\ &= C_1\, e^{2H_o t}\,\varphi_o(t)\,\exp(\mathrm{i}\vec{k}\cdot\vec{x})\,\mathcal{A}^2(y)\,\hat{e}_{ij}\,, \end{aligned} \quad (32)$$

where $\mathcal{A}(y)$ is given by Eq. (21). Thus the bulk dependence of this mode is the same as that found by Randall and Sundrum [1] for a Minkowski brane, with h_{ij} decaying away from the brane proportional to the 'warp factor', $\mathcal{A}^2$, which for the RS brane ($H_o \to 0$) is given by $\mathcal{A}^2 = e^{-2\mu|y|}$.

Introducing the conformal bulk coordinate z, given in Eq. (25), and defining $\Psi_m \equiv \mathcal{A}^{3/2}\mathcal{E}_m$, it is possible to rewrite the off-brane equation in the Schrödinger-like form

$$\frac{d^2\Psi_m}{dz^2} - V\Psi_m = -m^2\Psi_m\,, \quad (33)$$

where

$$\begin{aligned} V(z) &= \tfrac{15}{4}\mu^2\mathcal{A}^2(z) + \tfrac{9}{4}H_o^2 - 3\mu\left[1+\frac{\rho}{\lambda}\right]\delta(z-z_\mathrm{b})\,,\\ &= \frac{15H_o^2}{4\sinh^2(H_o z)} + \tfrac{9}{4}H_o^2 - 3\mu\left[1+\frac{\rho}{\lambda}\right]\delta(z-z_\mathrm{b})\,. \end{aligned} \quad (34)$$

In the RS limit, $\rho \to 0$ and $H_o \to 0$, so that $\mathcal{A}^2 \to [1+\mu(|z|-z_\mathrm{b})]^{-2}$. The effect in the off-brane equation of introducing curvature (expansion) on the brane is two-fold. Firstly it changes the form of $\mathcal{A}(z)$, so that it is no longer possible to obtain closed form solutions for the Kaluza-Klein (KK) modes for arbitrary m; however, the qualitative behaviour of $\mathcal{A}(z)$ remains the same, i.e., a cusp at the brane, dying off monotonically to 0 as $z \to \infty$. The second effect is more important, and is the introduction of a constant term into V, so that $V > 0$ at infinity. This can be treated as a shift in the effective mass-squared:

$$m^2 \to \hat{m}^2 = m^2 - m_\mathrm{c}^2\,, \quad m_\mathrm{c} = \tfrac{3}{2}H_o\,.$$

For $m > m_\mathrm{c}$, the KK modes for a de Sitter brane correspond qualitatively to those for a flat brane, asymptotically approaching sinusoidal functions $\Psi_m \sim \exp(\mathrm{i}\hat{m}|z|)$ as $|z| \to \infty$, but with the reduced effective mass, $\hat{m}$. For $m < m_\mathrm{c}$, the effective mass of the free field at infinity becomes negative, signalling an instability, and in general these modes will diverge at infinity and thus be non-normalizable. As remarked earlier, the zero-mode ($m = 0$) remains finite (and normalizable) due to the boundary condition at the brane.

IV. QUANTUM FLUCTUATIONS ON THE BRANE

A full treatment of the spectrum of graviton fluctuations generated by de Sitter inflation on the brane requires a physical prescription for the boundary condition on the Cauchy horizon at past null infinity, which in turn requires a quantum cosmological extension, as considered, for instance, in Refs. [17,18]. However we can gain some insight by simply treating each mode φ_m as a quantum field in four-dimensions with time-dependent potential, as is done in conventional 4-D inflation models, where we demand that the fields are in their adiabatic vacuum state at early-times/small-scales ($k \gg a_o H_o$) well within the horizon. The extra subtlety in our 5-D model comes from determining the orthonormal basis for the y-dependent functions and from the need to determine which of these eigenmodes to include.

As in Ref. [1], we will require that the modes be normalizable in the measure dz, i.e.,

$$2\int_{z_\mathrm{b}}^{z_\mathrm{r}} |\Psi_m^2|\, dz = 1\,, \quad (35)$$

where z is the canonical coordinate for the Schrödinger-like Eq. (33), z_b is the position of our brane and z_r is the position of a 'regulator brane' [1]. This restricts the allowed values of m to the discrete mode $m = 0$, and a discrete spectrum of modes with m such that the boundary condition, $E' = 0$, also holds at z_r [1]. As $z_\mathrm{r} \to \infty$ this discrete spectrum approaches a continuum of states for $m > m_\mathrm{c}$. The normalization for the continuum modes is principally determined by the asymptotic sinusoidal

form $\Psi_m \sim \exp(\mathrm{i}\hat{m}|z|)$ as $|z| \to \infty$. However, since the time-dependent fields φ_m are heavy during inflation for $m > m_{\rm c}$, these modes remain in the vacuum state. Because the amplitude of these long-wavelength modes decays, their power spectrum is strongly suppressed on super-horizon scales ($k \ll a_o H_o$).

Normalizing the zero-mode, $\Psi_o = C_1 \mathcal{A}^{3/2}$, requires

$$2\int_{z_{\rm b}}^{\infty} |\Psi_o^2| dz = 2\int_0^{y_{\rm h}} C_1^2 \mathcal{A}^2 dy = 1\,, \tag{36}$$

which gives

$$C_1 = \sqrt{\mu}\; F(H_o/\mu)\;,$$

where

$$F(x) = \left\{ \sqrt{1+x^2} - x^2 \ln\left[\frac{1}{x} + \sqrt{1+\frac{1}{x^2}}\right]\right\}^{-1/2}\,. \tag{37}$$

At low energies, i.e., $H_0 \ll \mu$ or $\rho \ll \lambda$, we have $F \approx 1$, and recover the RS normalization $C_1^2 \to \mu$. But as the Hubble rate, and thus the energy-scale, of inflation rises, the value of $|z|$ at the brane increases and the integral in Eq. (36) is over a smaller interval, so that the normalization constant, and hence the amplitude of the normalized zero-mode at the brane, grows. For $H_o \gg \mu$, or $\rho \gg \lambda$, we obtain

$$C_1 \approx \sqrt{\mu}\sqrt{\frac{3H_o}{2\mu}} \gg \sqrt{\mu}\,.$$

The second-order effective action for tensor perturbations is

$$S_{\rm g} = \frac{1}{8\kappa_5^2}\int dt\, d^3\vec{x}\, dy\, na^3 \times\left[\frac{1}{n^2}\dot{E}_{ij}\dot{E}^{ij} - \frac{1}{a^2}\partial_\ell E_{ij}\partial^\ell E^{ij} - E'_{ij}E^{ij\prime}\right]\,. \tag{38}$$

For the zero-mode, given in Eq. (32) we can integrate over the bulk coordinate y, using the normalization given by Eq. (36), and this gives

$$S_{\rm g} = \frac{1}{8\kappa_5^2}\sum_{+,\times}\int d\eta\, d^3\vec{k}\, a_o^2\left[\left(\frac{d\varphi_o}{d\eta}\right)^2 + k^2{\varphi_o}^2\right]\,,$$

where we have also integrated over all three-dimensional Fourier modes and summed the two polarization states. This has the standard form for a massless graviton in four-dimensional cosmology [19,20], apart from the overall factor $1/8\kappa_5^2$ instead of $1/8\kappa_4^2$. It follows that quantum fluctuations in each polarization, φ_o, have an amplitude of $2\kappa_5(H_o/2\pi)$ on super-horizon scales. Quantum fluctuations on the brane at $y=0$, where $\mathcal{E}_o = C_1$, thus have the typical amplitude

$$\langle{\bar{\varphi}_o}^2\rangle^{1/2} = C_1\langle{\varphi_o}^2\rangle^{1/2} = 2\kappa_4\left(\frac{H_o}{2\pi}\right)F\left(H_o/\mu\right)\,, \tag{39}$$

on super-horizon scales, where $F(H_o/\mu)$ is given by Eq. (37).

At low energies ($H_0 \ll \mu$), we have $F \approx 1$, and we recover the standard four-dimensional result for gravitational waves produced during inflation [20]. At high energies ($H_0 \gg \mu$), we have $F \approx \sqrt{3H_0/2\mu}$, so that

$$\langle{\bar{\varphi}_o}^2\rangle^{1/2} \approx 2\kappa_4\left(\frac{H_o}{2\pi}\right)\sqrt{\frac{3H_o}{2\mu}}\,. \tag{40}$$

The amplitude of gravitational waves produced during inflation at high-energies is thus much greater than the standard result in four-dimensional general relativity.

We are thus able to calculate the primordial spectrum of gravitational wave perturbations produced by a period of slow-roll inflation, modelled as quasi-de Sitter expansion on the brane.

V. CONCLUSIONS

We have discussed the evolution of gravitational wave (tensor-type) perturbations in a cosmological brane-world scenario. In general the evolution of the metric perturbations on the brane is coupled to the evolution in the bulk, and it is not in general possible to separate a zero-mode corresponding to a massless graviton on the brane, from the massive states. However we find two special cases in which it is possible. Firstly on large scales, where spatial gradients on the brane can be neglected, there is always a zero-mode corresponding to a homogeneous metric perturbation in the bulk, $E=$const. Secondly the equation of motion for the perturbations becomes separable (with the brane remaining fixed in the bulk) in the case of de Sitter inflation on the brane.

Modes are characterized by their eigenvalue, $-m^2$, which becomes the negative mass-squared of the corresponding four-dimensional fields. We find that there is a discrete normalizable massless mode, $m=0$, as in the case of a non-expanding brane [1]. There is also a continuum of massive KK modes for $m > \frac{3}{2}H_o$, where H_o is the brane Hubble rate [17]. This mass gap is a consequence of the expansion, and may be understood in terms of an effective temperature of de Sitter spacetime creating a threshold of excitation as suggested in [17].

During a period of quasi-de Sitter inflation on the brane, the continuum of heavy modes remains under-damped and hence these modes are strongly suppressed on large scales, remaining in their vacuum state. However the massless mode is over-damped and acquires a spectrum of classical perturbations on super-horizon scales. By integrating over the bulk coordinate in the five-dimensional effective action, we are able to treat

the light mode as an effective four-dimensional field. At low energies, the massless mode is confined close to the brane, and we recover the standard four-dimensional result that fluctuations in the massless graviton on large scales are given by the Hawking temperature, $H_o/2\pi$. At higher energies the massless mode extends into the five-dimensional bulk and the amplitude of fluctuations on the brane is enhanced compared with the usual four-dimensional result.

Inflation at high energy scales thus enhances the amplitude of super-horizon tensor perturbations that can contribute to CMB anisotropies on large angular scales. However, the amplitude of scalar perturbations is also enhanced at high energies, as shown in [7]. While at low energies we recover the standard result that the ratio of tensor to scalar modes is given by the slow-roll parameter $\epsilon = M_4^2 V'^2/(16\pi V^2)$ (where V is the slow-roll potential), at high energies this ratio becomes[†]

$$\frac{A_{\rm T}^2}{A_{\rm S}^2} = 6\epsilon \left(\frac{\lambda}{\rho}\right) ,$$

where λ is the constant brane tension and ρ the inflaton energy density (see [20] for the precise definition of $A_{\rm T}$ and $A_{\rm S}$). Thus the relative contribution of gravitational waves to the CMB anisotropies is suppressed for inflation at energy scales $\rho \gg \lambda$.

Having been stretched beyond the horizon during inflation, the massless mode remains constant until horizon re-entry during the subsequent radiation and matter-dominated eras. We have shown that there is always a zero-mode solution, on sufficiently large scales ($k/a_oH_o \to 0$) with $\mathcal{E}_o(y) =$ constant, $\varphi_o =$ constant. At re-entry the expansion is no longer quasi-de Sitter, and the massless mode can no longer be decoupled from massive graviton modes. It remains to be seen whether any significant mixing between the massless zero-mode and the massive states occurs at horizon entry. This could lead to very interesting effects if the lightest continuum modes have a mass of order the Hubble scale (as is the case during de Sitter expansion), which would lead to a form of weakly interacting (gravitationally coupled) dark matter with a very light mass. It is intriguing to note a recent discussion [21] of the possibility of solving the problem of excess small-scale structure in the cores of galaxies, by hypothesizing that the dark matter has a very small mass, of order the present Hubble scale.

[†]This corrects the high-energy limit of the result given in Eq. (24) of Ref. [7], where the amplitude of gravitational waves was assumed to be the same as the 4-D result, leading to an even smaller ratio at high energies.

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